Bounds on and Constructions of Unit Time-Phase Signal Sets

Cunsheng Ding, Keqin Feng, Rongquan Feng, Aixian Zhang

Abstract

Digital signals are complex-valued functions on $\mathbf{Z}_n$. Signal sets with certain properties are required in various communication systems. Traditional signal sets consider only the time distortion during transmission. Recently, signal sets against both the time and phase distortion have been studied, and are called *time-phase* signal sets. Several constructions of time-phase signal sets are available in the literature. There are a number of bounds on time signal sets (also called codebooks). They are automatically bounds on time-phase signal sets, but are bad bounds. The first objective of this paper is to develop better bounds on time-phase signal sets from known bounds on time signal sets. The second objective of this paper is to construct two series of time-phase signal sets, one of which is optimal.

Index Terms

Codebooks, digital signals, phase distortion, sequences, signal sets, time distortion, time-phase signal sets.

I. Introduction

Throuout this paper, let $n > 1$ be an integer, and let $\mathbf{Z}_n = \{0, 1, \cdots, n-1\}$. We define $\mathbb{H}_n = \mathbb{C}(\mathbf{Z}_n)$, the set of all complex-valued functions on $\mathbf{Z}_n$, which is a Hilbert space with the Hermitian product given by

$$\langle \phi, \varphi \rangle = \sum_{t \in \mathbf{Z}_n} \phi(t)\overline{\varphi(t)}.$$

The *norm* of $\phi \in \mathbb{H}_n$ is defined by

$$\|\phi\| = \sqrt{\langle \phi, \phi \rangle}.$$

Digital signals are complex-valued functions on $\mathbf{Z}_n$. They are also called sequences as the following mapping

$$\phi \mapsto (\phi(0), \phi(1), \cdots, \phi(n-1))$$

transfers a function $\phi \in \mathbb{H}_n$ into a sequence in $\mathbb{C}^n$. We identify the function ϕ with its sequence. A subset $\mathcal{S} \subset \mathbb{H}_n$ is called a *signal set*, and a *real signal set* if every signal in $\mathcal{S}$ is real-valued. Any $\phi \in \mathbb{H}_n$ is called a *unit signal* if $\|\phi\| = 1$. A subset $\mathcal{S}$ is said to be a unit signal set if every signal in $\mathcal{S}$ is a unit signal. Signal sets with certain properties are required in some communication systems [10], [14], [26].

During the transmission process, a signal φ might be distorted. Two basic types of distortion are the *time shift* $\varphi(t) \mapsto \mathbf{L}_\tau\varphi(t) = \varphi(t+\tau)$ and the *phase shift* $\varphi(t) \mapsto \mathbf{M}_w\varphi(t) = e^{\frac{2\pi i}{n}wt}\varphi(t)$, where $\tau, w \in \mathbf{Z}_n$. For certain applications, it is required that for every $\varphi \neq \phi \in \mathcal{S}$,

$$|\langle \phi, \mathbf{M}_w\mathbf{L}_\tau\varphi \rangle| \ll 1. \tag{1}$$

C. Ding is with the Department of Computer Science and Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. Email: cding@ust.hk

K. Feng is with the Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P. R. China. Email: kfeng@math.tsinghua.edu.cn

R. Feng is with the School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China. Email: fengrq@math.pku.edu.cn

A. Zhang is with the School of Mathematical Sciences, Capital Normal University, Beijing,100048, P. R. China. Email: zhangaixian1008@126.com

In addition, signals are sometimes required to admit low peak-to-average power ratio, i.e., for every $\phi \in \mathcal{S}$ with $\|\phi\| = 1$,

$$\max\{|\phi(t)| : t \in \mathbf{Z}_n\} \ll 1.$$

In view of the above requirements and that every signal can be normalized into a unit signal, in this paper we consider only *unit signal sets* $\mathcal{S}$, in which we have for every unit signal ϕ

$$\frac{1}{\sqrt{n}} \leq \max\{|\phi(t)| : t \in \mathbf{Z}_n\}.$$

To measure the capability of anti-distortion of a signal set $\mathcal{S}$ with respect to the time and phase shift, Gurevich, Hadani and Sochen [10] defined

$$\lambda_{\mathcal{S}} = \max\{|\langle \phi, \mathbf{M}_w \mathbf{L}_\tau \varphi \rangle| : \text{ either } \phi \neq \varphi \text{ or } (\tau, w) \neq (0,0)\}. \tag{2}$$

For convenience, we call $\mathcal{S}$ an $(n, M, \lambda_{\mathcal{S}})$ *time-phase signal set*, where M denotes the total number of signals in $\mathcal{S}$. If we require that $\lambda_{\mathcal{S}} < 1$, any $(n, M, \lambda_{\mathcal{S}})$ time-phase signal set is an *ambiguity signal set* defined in [26].

In some communication systems, only time distortion is considered. In this case, two correlation measures are considered. One is the maximum crosscorrelation amplitude $\nu_{\mathcal{S}}$ of an (n, M) signal set defined by

$$\nu_{\mathcal{S}} = \max_{\substack{\phi, \varphi \in \mathcal{S} \\ \phi \neq \varphi}} |\langle \phi, \varphi \rangle|, \tag{3}$$

and the other is the maximum auto-and-cross correlation amplitude $\theta_{\mathcal{S}}$ of an (n, M) signal set defined by

$$\theta_{\mathcal{S}} = \max\{|\langle \phi, \mathbf{L}_\tau \varphi \rangle| : \text{ either } \phi \neq \varphi \text{ or } \tau \neq 0\}. \tag{4}$$

If only time distortion is considered, we call $\mathcal{S}$ an $(n, M, \nu_{\mathcal{S}})$ or $(n, M, \theta_{\mathcal{S}})$ *time signal set*. Time signal sets are also called *codebooks*.

By definition, we have clearly

$$\lambda_{\mathcal{S}} \geq \theta_{\mathcal{S}} \geq \nu_{\mathcal{S}} \tag{5}$$

for any (n, M) signal set.

Hence, signal sets are classified into two types: time-phase signal sets and time signal sets. Time signal sets have been studied for CDMA communications (see, for example, [1], [4], [5], [6], [8], [11], [14], [18], [21], [24], [27], [12]). A number of lower bounds on $\theta_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$ were developed. They are automatically lower bounds on $\lambda_{\mathcal{S}}$ due to (5), but are bad lower bounds for $\lambda_{\mathcal{S}}$ as the correlation measure $\lambda_{\mathcal{S}}$ is much stronger than $\theta_{\mathcal{S}}$ and $\nu_{\mathcal{S}}$. The objectives of this paper are to derive better lower bounds on the parameters of time-phase signal sets, and construct optimal and good optimal time-phase signal sets.

This paper is organized as follows. Section II first establishes a one-way bridge between time-phase signal sets and time signal sets, and then uses this bridge to develop bounds on unit time-phase signal sets from known bounds on time signal sets. Section III first sets up another one-way bridge between time-phase signal sets and time signal sets, and then employs this bridge to develop more bounds on unit time-phase signal sets from known bounds on time signal sets. Section IV presents two series of unit time-phase signal sets, one of which is optimal. Section V summaries the main contributions of this paper and presents some open problems.

II. The first group of lower bounds on (n, M, λ) time-phase signal sets

Two bounds on the parameters of $(n, M, \nu_{\mathcal{S}})$ time signal sets are described in the following two lemmas.

Lemma 1: (Welch's bound [27]) For any (n, M, ν) unit time signal set $\mathcal{S}$ with $M \geq n$, and for each integer $k \geq 1$,

$$\sum_{\phi,\varphi\in\mathcal{S}} |\langle\phi,\varphi\rangle|^{2k} \geq \binom{n+k-1}{k}^{-1} M^2,$$

so that

$$\begin{aligned}\nu_{\mathcal{S}} &\geq \left(\frac{1}{M(M-1)}\sum_{\substack{\phi,\varphi\in\mathcal{S}\\ \phi\neq\varphi}} |\langle\phi,\varphi\rangle|^{2k}\right)^{1/2k} \\ &\geq \left(\frac{\binom{n+k-1}{k}^{-1}M^2-M}{M(M-1)}\right)^{1/2k} \\ &= \left(\frac{M-\binom{n+k-1}{k}}{(M-1)\binom{n+k-1}{k}}\right)^{1/2k} \triangleq \widetilde{w_k},\end{aligned} \tag{6}$$

and $\nu_{\mathcal{S}} = \widetilde{w_k}$ if and only if for all pairs $(\phi,\varphi) \in \mathcal{S}\times\mathcal{S}$ with $\phi \neq \varphi, |\langle\phi,\varphi\rangle| = \widetilde{w_k}$.

It was proved in [24] that no (n, M, ν) real time signal set $\mathcal{S}$ can meet the Welch bound $\widetilde{w_1}$ of (6) if $M > n(n+1)/2$ and no (n, M, ν) time signal set $\mathcal{S}$ can meet the Welch bound $\widetilde{w_1}$ of (6) if $M > n^2$.

The following was proved in [17]

Lemma 2: For any $(n, M, \nu_{\mathcal{S}})$ real unit time signal set $\mathcal{S}$ with $M > n(n+1)/2$,

$$\nu_{\mathcal{S}} \geq \sqrt{\frac{3M-n^2-2n}{(n+2)(M-n)}}. \tag{7}$$

For any $(n, M, \nu_{\mathcal{S}})$ complex unit time signal set $\mathcal{S}$ with $M > n^2$,

$$\nu_{\mathcal{S}} \geq \sqrt{\frac{2M-n^2-n}{(n+1)(M-n)}}. \tag{8}$$

If $M < n(n+1)/2$ (respectively $M < n^2$), the Welch bound $\widetilde{w_1} = \sqrt{\frac{M-n}{(M-1)n}}$ on real (respectively, complex) time signal sets is better. However, the Levenstein bound of (7) on real time signal sets is tighter than Welch's bound $\widetilde{w_1}$ if $M > n(n+1)/2$, and that the Levenstein bound of (8) on complex time signal sets is tighter than Welch's bound $\widetilde{w_1}$ if $M > n^2$.

Welch's and Levenstein's lower bounds on ν for time signal sets yield directly lower bounds on λ for time-phase signal sets according to (5). But they are very bad lower bounds on λ as the correlation measure $\lambda_{\mathcal{S}}$ is much stronger than $\nu_{\mathcal{S}}$ and $\theta_{\mathcal{S}}$. However, they can be employed to derive better bounds on $\lambda_{\mathcal{S}}$ for time-phase signal sets. This is our task in this section.

Lemma 3: For any pair of distinct signals ϕ and φ in an (n, M, λ) unit time-phase signal set with $\lambda < 1$, we have

$$\phi \neq \mathbf{M}_w\mathbf{L}_\tau\varphi$$

for any $w, \tau \in \mathbf{Z}_n$.

Proof: The conclusion follows directly from the assumption that $\lambda < 1$. ■

Given an (n, M, λ) unit time-phase signal set $\mathcal{S}$ with $\lambda < 1$, where

$$\mathcal{S} = \{\phi_1, \phi_2, \cdots, \phi_M\},$$

we define an (n, n^2M, λ) unit time signal set

$$\mathcal{S}_{\mathcal{S}} = \{\phi_{j,w,\tau} : 1 \le j \le M,\ 0 \le w \le n-1,\ 0 \le \tau \le n-1\}, \tag{9}$$

where

$$\phi_{j,w,\tau}(t) = e^{\frac{2\pi i}{n}wt}\phi_j(t+\tau). \tag{10}$$

Lemma 4: Let $\phi_{j,w,\tau}$ be defined in (10). Then each $\phi_{j,w,\tau}$ is a unit signal. In addition $\phi_{j,w,\tau} = \phi_{j',w',\tau'}$ if and only if $(j,w,\tau) = (j',w',\tau')$.

Proof: The first conclusion is straightforward, and the second follows from Lemma 3. ∎

Theorem 5: For any $(n, M, \lambda_{\mathcal{S}})$ unit time-phase signal set $\mathcal{S}$ with $\lambda_{\mathcal{S}} < 1$, the set $\mathcal{S}_{\mathcal{S}}$ defined in (9) is an $(n, n^2M, \nu_{\mathcal{S}_{\mathcal{S}}})$ unit time signal set with $\nu_{\mathcal{S}_{\mathcal{S}}} = \lambda_{\mathcal{S}}$.

Proof: It follows from Lemma 4 that $|\mathcal{S}_{\mathcal{S}}| = n^2M$. By definition,

$$\begin{aligned}
|\langle \phi_{j,w,\tau}, \phi_{j',w',\tau'}\rangle| &= \left|\sum_{t=0}^{n-1} e^{\frac{2\pi i}{n}wt}\phi_j(t+\tau)e^{-\frac{2\pi i}{n}w't}\overline{\phi_{j'}(t+\tau')}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t+\tau)e^{-\frac{2\pi i}{n}(w'-w)t}\overline{\phi_{j'}(t+\tau')}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t)e^{-\frac{2\pi i}{n}(w'-w)(t-\tau)}\overline{\phi_{j'}(t+\tau'-\tau)}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t)e^{-\frac{2\pi i}{n}(w'-w)t}\overline{\phi_{j'}(t+\tau'-\tau)}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t)\overline{e^{\frac{2\pi i}{n}(w'-w)t}\phi_{j'}(t+\tau'-\tau)}\right| \\
&= \left|\langle \phi_j, \mathbf{M}_{(w'-w) \bmod n}\mathbf{L}_{(\tau'-\tau) \bmod n}\phi_{j'}\rangle\right|.
\end{aligned}$$

Note that $(w'-w) \bmod n$ ranges over all elements in $\mathbf{Z}_n$ when both w and w' run over all elements in $\mathbf{Z}_n$. The same is true for $(\tau'-\tau) \bmod n$. Hence, the signal set $\mathcal{S}_{\mathcal{S}}$ has maximum crosscorrelation amplitude $\nu_{\mathcal{S}_{\mathcal{S}}} = \lambda_{\mathcal{S}}$. ∎

Remark 1: Theorem 5 is one of the main contributions of this paper. It serves as a one-way bridge with which bounds on time signal sets can be employed to derive better lower bounds on time-phase signal sets.

Theorem 6: For any $(n, M, \lambda_{\mathcal{S}})$ unit time-phase signal set $\mathcal{S}$ with $\lambda_{\mathcal{S}} < 1$ and each integer $k \ge 1$, we have

$$\lambda_{\mathcal{S}} \ge w_k \triangleq \left(\frac{n^2M - \binom{n+k-1}{k}}{(n^2M-1)\binom{n+k-1}{k}}\right)^{1/2k}. \tag{11}$$

Proof: The desired conclusion follows from Lemma 1 and Theorem 5. ∎

Theorem 7: For any $(n, M, \lambda_{\mathcal{S}})$ unit time-phase signal set $\mathcal{S}$ with $\lambda < 1$ and $M > 1$,

$$\lambda_{\mathcal{S}} \ge \sqrt{\frac{2nM - n - 1}{(n+1)(nM-1)}}. \tag{12}$$

Proof: The desired conclusion follows from Lemma 2 and Theorem 5. ∎

Remark 2: For any (n, M, λ) unit time-phase signal set $\mathcal{S}$ with $M > n^2$, the Levenstein bound gives automatically the following bound:

$$\lambda_{\mathcal{S}} \ge \sqrt{\frac{2M - n^2 - n}{(n+1)(M-n)}}.$$

However, it can be checked that this lower bound is much smaller than the lower bound of (12) when $M > n^2$. This shows that the new bound of (12) is indeed a big improvement over the original Levenstein bound when it is used as a bound for unit time-phase signal sets. This comment also applies to the new bound of Theorem 6.

Remark 3: It is easy to verify that the bounds w_1 of (11) and (12) are the same when $M = 1$, and the latter is superior when $M > 1$. So the bound of (11) is useful only for the case that $M = 1$. The purpose of presenting Theorem 6 here is to show that the Levenstein bound yields better bound on unit time-phase signal sets under the framework of this section. We will construct some series of unit time-phase signal sets in Section IV and then discuss the tightness of these bounds in Section V.

III. The second group of bounds on (n, M, λ) time-phase signal sets

In this section, we further derive bounds on time-phase unit signal sets from known bounds on time unit signal sets.

Given an $(n, M, \lambda_{\mathcal{S}})$ time-phase unit signal set $\mathcal{S}$ with $\lambda_{\mathcal{S}} < 1$, where

$$\mathcal{S} = \{\phi_1, \phi_2, \cdots, \phi_M\},$$

we now define an (n, nM) unit time signal set

$$\bar{\mathcal{S}}_{\mathcal{S}} = \{\phi_{j,w} : 1 \le j \le M,\ 0 \le w \le n-1\}, \tag{13}$$

where

$$\phi_{j,w}(t) = e^{\frac{2\pi i}{n} wt}\phi_j(t). \tag{14}$$

Lemma 8: Let $\phi_{j,w}$ be defined in (14). Then each $\phi_{j,w}$ is a unit signal. In addition, $\phi_{j,w} = \phi_{j',w'}$ if and only if $(j, w) = (j', w')$.

Proof: The first conclusion is straightforward, and the second one follows from Lemma 3. ∎

Theorem 9: For any $(n, M, \lambda_{\mathcal{S}})$ unit time-phase signal set $\mathcal{S}$ with $\lambda_{\mathcal{S}} < 1$, the set $\bar{\mathcal{S}}_{\mathcal{S}}$ defined in (13) is an $(n, nM, \theta_{\bar{\mathcal{S}}_{\mathcal{S}}})$ unit time signal set with $\theta_{\bar{\mathcal{S}}_{\mathcal{S}}} = \lambda_{\mathcal{S}}$.

Proof: It follows from Lemma 8 that $\left|\bar{\mathcal{S}}_{\mathcal{S}}\right| = nM$. By definition,

$$\begin{aligned}
|\langle \phi_{j,w}, \mathbf{L}_\tau \phi_{j',w'}\rangle| &= \left|\sum_{t=0}^{n-1} e^{\frac{2\pi i}{n} wt}\phi_j(t) e^{-\frac{2\pi i}{n} w'(t+\tau)}\overline{\phi_{j'}(t+\tau)}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t) e^{-\frac{2\pi i}{n}(w'-w)t}\overline{\phi_{j'}(t+\tau)}\right| \\
&= \left|\sum_{t=0}^{n-1} \phi_j(t) \overline{e^{\frac{2\pi i}{n}(w'-w)t}\phi_{j'}(t+\tau)}\right| \\
&= \left|\langle \phi_j, \mathbf{M}_{(w'-w) \bmod n}\mathbf{L}_\tau \phi_{j'}\rangle\right|.
\end{aligned}$$

Note that $(w'-w) \bmod n$ ranges over all elements in $\mathbf{Z}_n$ when both w and w' run over all elements in $\mathbf{Z}_n$. Hence, for the signal set $\bar{\mathcal{S}}_{\mathcal{S}}$ we have $\theta_{\bar{\mathcal{S}}_{\mathcal{S}}} = \lambda_{\mathcal{S}}$. ∎

Remark 4: Theorem 9 is another main contribution of this paper. It serves as a one-way bridge with which bounds on time signal sets can be employed to derive better lower bounds on time-phase signal sets.

The following bounds on unit time signal sets are due to Levenstein [17], [12], and are linear programming bounds. They work automatically as bounds for unit time-phase signal sets, but are bad ones because the bounds of Theorem 11 are much better.

Lemma 10: Let $\mathcal{S} \subset \mathbb{H}_n$ be any (n, M, θ) unit time signal set. Then

$$M \leq \begin{cases} \frac{1-\theta^2}{1-n\theta^2}, & \text{if } 0 < \theta^2 \leq \frac{1}{n+1} \\ \frac{(n+1)(1-\theta^2)}{2-(n+1)\theta^2}, & \text{if } \frac{1}{n+1} < \theta^2 \leq \frac{2}{n+2} \\ \frac{n(n+1)(n+2)(1-\theta^2)^2}{(n+1)(n+2)\theta^4-4(n+1)\theta^2+2}, & \text{if } \frac{2}{n+2} < \theta^2 \leq \frac{2(n+2)+\sqrt{2(n+1)(n+2)}}{(n+2)(n+3)} \\ \frac{n(n+1)(n+2)[(n+3)\theta^2-2](1-\theta^2)}{12(n+2)\theta^2-2(n+2)(n+3)\theta^4-12}, & \text{if } \frac{2(n+2)+\sqrt{2(n+1)(n+2)}}{(n+2)(n+3)} \leq \theta^2 \leq \frac{3(n+3)+\sqrt{3(n+3)(n+1)}}{(n+3)(n+4)}. \end{cases}$$

The bounds on unit time-phase signal sets described in the following theorem are derived from the bounds of Lemma 10 and are better.

Theorem 11: Let $\mathcal{S} \subset \mathbb{H}_n$ be any (n, M, λ) unit time-phase signal set. Then

$$nM \leq \begin{cases} \frac{1-\lambda^2}{1-n\lambda^2}, & \text{if } 0 < \lambda^2 \leq \frac{1}{n+1} \\ \frac{(n+1)(1-\lambda^2)}{2-(n+1)\lambda^2}, & \text{if } \frac{1}{n+1} < \lambda^2 \leq \frac{2}{n+2} \\ \frac{n(n+1)(n+2)(1-\lambda^2)^2}{(n+1)(n+2)\lambda^4-4(n+1)\lambda^2+2}, & \text{if } \frac{2}{n+2} < \lambda^2 \leq \frac{2(n+2)+\sqrt{2(n+1)(n+2)}}{(n+2)(n+3)} \\ \frac{n(n+1)(n+2)[(n+3)\lambda^2-2](1-\lambda^2)}{12(n+2)\lambda^2-2(n+2)(n+3)\lambda^4-12}, & \text{if } \frac{2(n+2)+\sqrt{2(n+1)(n+2)}}{(n+2)(n+3)} \leq \lambda^2 \leq \frac{3(n+3)+\sqrt{3(n+3)(n+1)}}{(n+3)(n+4)}. \end{cases} \tag{15}$$

Proof: The desired conclusions of this theorem follow from Theorem 9 and Lemma 10. ■

Remark 5: The first bound in (15) coincides with the bound of (11) in the case $k = 1$. The second bound in (15) coincides with the bound of (12). So these bounds can be derived with both bridges established in this paper.

To introduce more bounds on time-phase unit signal sets, let $\mathbb{H}_{(n,q)}$ denote the set of all complex-valued functions f on $\mathbf{Z}_n$ such that $\sqrt{n}f(i)$ is a qth root of unity for all $i \in \mathbf{Z}_n$.

The following bounds on unit time signal sets are due to Levenstein [17], [12], and are linear programming bounds. They work automatically as bounds for unit time-phase signal sets, but are bad ones because the bounds of Theorem 13 are much better.

Lemma 12: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, θ) unit time signal set, where $q = 2$. Then

$$M \leq \begin{cases} \frac{1-\theta^2}{1-n\theta^2}, & \text{if } 0 \leq \theta^2 \leq \frac{n-2}{n^2} \\ \frac{n^2(1-\theta^2)}{3n-2-n^2\theta^2}, & \text{if } \frac{n-2}{n^2} \leq \theta^2 \leq \frac{3n-8}{n^2} \\ \frac{n(1-\theta^2)[(n-2)(n^2-3n+8)-(n^2-n+2)n^2\theta^2]}{6n(n-2)-4(3n-4)n^2\theta^2+2n^4\theta^4}, & \text{if } \frac{3n-8}{n^2} \leq \theta^2 \leq \frac{3n-10+\sqrt{6n^2-42n+76}}{n^2} \\ \frac{n^2(1-\theta^2)}{6}\frac{3n^3-23n^2+90n-136-(n^2-3n+8)n^2\theta^2}{15n^2-50n+24-10(n-2)n^2\theta^2+n^4\theta^4}, & \text{if } \frac{3n-10+\sqrt{6n^2-42n+76}}{n^2} \leq \theta^2 \leq \frac{5(n-4)+\sqrt{10n^2-90n+216}}{n^2}. \end{cases}$$

The bounds on unit time-phase signal sets described in the following theorem are derived from the bounds of Lemma 12 and are better.

Theorem 13: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, λ) unit time-phase signal set, where $q = 2$. Then

$$nM \leq \begin{cases} \frac{1-\lambda^2}{1-n\lambda^2}, & \text{if } 0 \leq \lambda^2 \leq \frac{n-2}{n^2} \\ \frac{n^2(1-\lambda^2)}{3n-2-n^2\lambda^2} & \text{if } \frac{n-2}{n^2} \leq \lambda^2 \leq \frac{3n-8}{n^2} \\ \frac{n(1-\lambda^2)[(n-2)(n^2-3n+8)-(n^2-n+2)n^2\theta^2]}{6n(n-2)-4(3n-4)n^2\theta^2+2n^4\theta^4} & \text{if } \frac{3n-8}{n^2} \leq \lambda^2 \leq \frac{3n-10+\sqrt{6n^2-42n+76}}{n^2} \\ \frac{n^2(1-\lambda^2)}{6}\frac{3n^3-23n^2+90n-136-(n^2-3n+8)n^2\theta^2}{15n^2-50n+24-10(n-2)n^2\theta^2+n^4\theta^4} & \text{if } \frac{3n-10+\sqrt{6n^2-42n+76}}{n^2} \leq \lambda^2 \leq \frac{5(n-4)+\sqrt{10n^2-90n+216}}{n^2}. \end{cases} \tag{16}$$

Proof: The desired conclusions of this theorem follow from Theorem 9 and Lemma 12. ■

Remark 6: The second bound in (16) is better than the second bound in (15) when $n > 2$. But the former applies only to binary real time-phase signal sets, while the latter applies to all time-phase signal sets.

The following bounds on unit time signal sets are due to Levenstein [17], [12], and are linear programming bounds. They work automatically as bounds for unit time-phase signal sets, but are bad ones because the bounds of Theorem 15 are much better.

Lemma 14: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, θ) unit time signal set, where $q \geq 3$. Then

$$M \leq \begin{cases} \frac{1-\theta^2}{1-n\theta^2}, & \text{if } 0 \leq \theta^2 \leq \frac{n-1}{n^2} \\ \frac{n^2(1-\theta^2)}{2n-1-n^2\theta^2}, & \text{if } \frac{n-1}{n^2} \leq \theta^2 \leq \frac{2n^2-5n+4}{n^2(n-1)} \\ \frac{n^2(1-\theta^2)[(n^2-n+1)n^2\theta^2-n^3+3n^2-5n+4]}{n[4(n-1)n^2\theta^2-n^4\theta^4-2n^2+3n]}, & \text{if } \frac{2n^2-5n+4}{n^2(n-1)} \leq \theta^2 \leq \frac{2n-2+\sqrt{2n^2-5n+4}}{n^2}. \end{cases}$$

The bounds on unit time-phase signal sets described in the following theorem are derived from the bounds of Lemma 14 and are better.

Theorem 15: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, λ) unit time-phase signal set, where $q \geq 3$. Then

$$nM \leq \begin{cases} \frac{1-\lambda^2}{1-n\lambda^2}, & \text{if } 0 \leq \lambda^2 \leq \frac{n-1}{n^2} \\ \frac{n^2(1-\lambda^2)}{2n-1-n^2\lambda^2}, & \text{if } \frac{n-1}{n^2} \leq \lambda^2 \leq \frac{2n^2-5n+4}{n^2(n-1)} \\ \frac{n^2(1-\lambda^2)[(n^2-n+1)n^2\theta^2-n^3+3n^2-5n+4]}{n[4(n-1)n^2\theta^2-n^4\lambda^4-2n^2+3n]}, & \text{if } \frac{2n^2-5n+4}{n^2(n-1)} \leq \lambda^2 \leq \frac{2n-2+\sqrt{2n^2-5n+4}}{n^2}. \end{cases}$$

Proof: The desired conclusions of this theorem follow from Theorem 9 and Lemma 14. ∎

The following bounds on unit time signal sets are due to Sidelnikov [23]. They work automatically as bounds for unit time-phase signal sets, but are bad ones because the bounds of Theorem 17 are much better.

Lemma 16: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, θ) unit time signal set. Then

$$\theta^2 \geq \begin{cases} \frac{(2k+1)(n-k)}{n^2} + \frac{k(k+1)}{2n^2} - \frac{2^k n^{2k}}{M(2k)!\binom{n}{k}} & \text{if } 0 \leq k \leq \frac{2n}{5} \text{ and } q = 2 \\ \frac{(k+1)(2n-k)}{2n^2} - \frac{2^k n^{2k}}{M(k!)^2\binom{n}{k}} & \text{if } k \geq 0 \text{ and } q > 2. \end{cases}$$

The bounds on unit time-phase signal sets described in the following theorem are derived from the Sidelnikov bounds of Lemma 16 and are better.

Theorem 17: Let $\mathcal{S} \subset \mathbb{H}_{(n,q)}$ be any (n, M, λ) unit time-phase signal set. Then

$$\lambda^2 \geq \begin{cases} \frac{(2k+1)(n-k)}{n^2} + \frac{k(k+1)}{2n^2} - \frac{2^k n^{2k}}{nM(2k)!\binom{n}{k}} & \text{if } 0 \leq k \leq \frac{2n}{5} \text{ and } q = 2 \\ \frac{(k+1)(2n-k)}{2n^2} - \frac{2^k n^{2k}}{nM(k!)^2\binom{n}{k}} & \text{if } k \geq 0 \text{ and } q > 2. \end{cases}$$

Proof: The desired conclusions of this theorem follow from Theorem 9 and Lemma 16. ∎

IV. CONSTRUCTIONS OF UNIT TIME-PHASE SIGNAL SETS

In this section we present two series of unit time-phase signal sets which are related to Gaussian sums. We first introduce some basic facts on Gaussian sums that will be employed in this section. For more information the reader is referred to [3].

Let $\zeta_n = e^{\frac{2\pi\sqrt{-1}}{n}}$ $(n \geq 2)$, $q = p^m$ where $m \geq 1$ and p is a prime number. Let $\mathrm{T} : \mathbb{F}_q \to \mathbb{F}_p$ be the trace mapping. The group of additive characters of $(\mathbb{F}_q, +)$ is

$$\widehat{\mathbb{F}}_q = \{\psi_b : b \in \mathbb{F}_q\},$$

where $\psi_b : \mathbb{F}_q \to \langle\zeta_p\rangle$ is defined by

$$\psi_b(x) = \zeta_p^{\mathrm{T}(bx)} \quad (x \in \mathbb{F}_q). \tag{17}$$

The identity (trivial character) is $\psi_0 = 1$ and the inverse of ψ_b is ψ_{-b}.

Let γ be a primitive element of $\mathbb{F}_q$ so that

$$\mathbb{F}_q^\times = \mathbb{F}_q \backslash \{0\} = \{1, \gamma, \gamma^2, \cdots, \gamma^{q-2}\}.$$

The group of multiplicative characters of $\mathbb{F}_q$ is

$$(\mathbb{F}_q^\times)^\wedge = \langle\omega\rangle = \{\omega^i : 0 \leq i \leq q-2\},$$

where $\omega : \mathbb{F}_q^\times \to \langle \zeta_{q-1} \rangle$ is defined by

$$\omega(\gamma^j) = \zeta_{q-1}^j \quad (0 \le j \le q-2).$$

The identity (trivial character) is $\omega^0 = 1$ and the inverse of ω^i is the conjugate character $\bar{\omega}^i = \omega^{-i}$.

For an additive character ψ and multiplicative character χ of $\mathbb{F}_q$, the Gauss sum over $\mathbb{F}_q$ is defined by

$$G(\psi, \chi) = \sum_{x \in \mathbb{F}_q^\times} \psi(x)\chi(x) \tag{18}$$

Lemma 18: Let ψ and χ be an additive and multiplicative character of $\mathbb{F}_q$ respectively. Then

$$G(\psi, \chi) = \begin{cases} q-1, & \text{if } \psi = 1 \text{ and } \chi = 1; \\ -1, & \text{if } \psi \neq 1 \text{ and } \chi = 1; \\ 0, & \text{if } \psi = 1 \text{ and } \chi \neq 1. \end{cases}$$

If $\psi = \psi_b \neq 1$ (namely, $b \neq 0$) and $\chi \neq 1$, then

$$|G(\psi, \chi)| = \sqrt{q},$$

and

$$G(\psi_b, \chi) = \bar{\chi}(b) G(\chi),$$

where

$$G(\chi) = G(\psi_1, \chi) = \sum_{x \in \mathbb{F}_q^\times} \psi_1(x)\chi(x) = \sum_{x \in \mathbb{F}_q^\times} \chi(x)\zeta_p^{T(x)}.$$

The following theorem describes a infinite series of unit time-phase signal sets for the case $M = 1$.

Theorem 19: Let $q = p^l, n = q-1, \mathrm{T} : \mathbb{F}_q \to \mathbb{F}_p$ be the trace mapping, γ be a primitive element of $\mathbb{F}_q$. Let

$$\phi = \frac{1}{\sqrt{n}}(\phi(0), \phi(1), \cdots, \phi(n-1)) \in \mathbb{C}^n$$

where

$$\phi(i) = \zeta_p^{T(\gamma^i)} \quad (0 \le i \le n-1).$$

Then $\mathcal{S} = \{\phi\}$ is an $(n, 1, \frac{\sqrt{n+1}}{n})$ unit time-phase signal set.

Proof: For $(w, \tau) \neq (0, 0),\ 0 \le w, \tau \le n-1$

$$\begin{aligned} \langle \phi, \mathbf{M}_w \mathbf{L}_\tau(\phi) \rangle &= \frac{1}{n} \sum_{i=0}^{n-1} \zeta_p^{T(\gamma^i)} \bar{\zeta}_p^{T(\gamma^{i+\tau})} \bar{\zeta}_n^{iw} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \zeta_p^{T(\gamma^i(1-\gamma^\tau))} \bar{\zeta}_n^{iw} \\ &= \frac{1}{n} \sum_{x \in \mathbb{F}_q^\times} \psi_b(x) \chi^w(x) = \frac{1}{n} G(\psi_b, \chi^w) \end{aligned} \tag{19}$$

where $b = 1 - \gamma^\tau$ and ψ_b is the additive character of $\mathbb{F}_q$ defined by (17), χ is the multiplicative character of $\mathbb{F}_q^\times$ defined by $\chi(\gamma) = \bar{\zeta}_n$ and $G(\psi_b, \chi^w)$ is the Gauss sum defined by (18). From Lemma 18 we have

$$|\langle \phi, \mathbf{M}_w \mathbf{L}_\tau(\phi) \rangle| = \begin{cases} \frac{1}{n}, & \text{if } \ \tau = 0 \text{ (so that } b = 0) \text{ and } w \neq 0, \\ 0, & \text{if } \tau \neq 0 \text{ (so that } b \neq 0) \text{ and } w = 0, \\ \frac{1}{n}|G(\psi_b, \chi^w)| = \frac{\sqrt{n+1}}{n}, & \text{if } \ \tau \neq 0 \text{ and } w \neq 0. \end{cases}$$

Therefore $\lambda = \frac{\sqrt{n+1}}{n}$. ■

Remark 7: In the case that $M=1$, the bound of (11) is $1/\sqrt{n+1}$. The existence of an $(n,M,\lambda)=(n,1,1/\sqrt{n+1})$ unit time-phase signal set for any $n\geq 2$ is equivalent to a particular kind of SIC-POVM in quantum information theory ([20], [30]). But so far only for finitely many of n such a SIC-POVM has been constructed [22]. In other words, no infinite series of $(n,1,1/\sqrt{n+1})$ unit time-phase signal sets are known. The infinite series of unit time-phase signal sets of Theorem 19 almost meet the bound of (11).

When $q\geq 3$ and $\frac{n-1}{n^2}\leq\lambda^2\leq\frac{2n^2-5n+4}{n^2(n-1)}$, the second bound of Theorem 15 becomes

$$M\leq\left\lfloor\frac{n(1-\lambda^2)}{2n-1-n^2\lambda^2}\right\rfloor.$$

It is easily verified that the infinite series of unit time-phase signal sets of Theorem 19 meet this bound, and are thus optimal. This is the first time that an infinite series of optimal time-phase signal sets is constructed.

From now on we assume $M\geq 2$. In this case it is easy to see that the lower bound $w_2=(\frac{2Mn-(n+1)}{(n+1)(Mn^2-1)})^{1/4}$ is tighter than $w_1=1/\sqrt{n+1}$ and the bound $\sqrt{\frac{2nM-n-1}{(n+1)(nM-1)}}$ is tighter than w_2 for all $M\geq 2$.

Now we present a cyclotomic construction of unit time-phase signal sets for $M\geq 2$. The construction is a generalization of Theorem 19.

Theorem 20: Let $q=p^l,\ q-1=en\ (e\geq 2),\ T:\mathbb{F}_q\to\mathbb{F}_p$ be the trace mapping, γ be a primitive element of $\mathbb{F}_q$. For $0\leq i\leq e-1$, let

$$\phi_i=\frac{1}{\sqrt{n}}(\phi_i(0),\phi_i(1),\cdots,\phi_i(n-1))\in\mathbb{C}^n$$

where

$$\phi_i(l)=\zeta_p^{T(\gamma^{i+le})}\quad(0\leq l\leq n-1).$$

Then $\mathcal{S}=\{\phi_i:0\leq i\leq e-1\}$ is an (n,M,λ) unit time-phase signal set where $n=\frac{q-1}{e},M=e$ and $\lambda\leq\frac{\sqrt{en+1}}{n}$.

Proof: For $0\leq i,j,w,\tau\leq n-1,\ (i-j,w,\tau)\neq(0,0,0)$,

$$\begin{aligned}\langle\phi_i,\mathbf{M}_w\mathbf{L}_\tau(\phi_j)\rangle &= \frac{1}{n}\sum_{l=0}^{n-1}\zeta_p^{T(\gamma^{i+le})}\bar{\zeta}_p^{T(\gamma^{j+(l+\tau)e})}\bar{\zeta}_n^{lw}\\ &= \frac{1}{n}\sum_{l=0}^{n-1}\zeta_p^{T(\gamma^{le}(\gamma^i-\gamma^{j+\tau e}))}\bar{\zeta}_n^{lw}\\ &= \frac{1}{n}\sum_{l=0}^{n-1}\zeta_p^{T(\beta\gamma^{le})}\chi^w(\gamma^{le}) \qquad (20)\end{aligned}$$

where χ is the multiplicative character of $\mathbb{F}_q^\times$ defined by $\chi(\gamma)=\bar{\zeta}_{q-1}$ and $\beta=\gamma^i-\gamma^{j+\tau e}$. Since for $\gamma^t,\ 0\leq t\leq q-2$,

$$\sum_{s=0}^{e-1}\chi^{ns}(\gamma^t)=\sum_{s=0}^{e-1}\bar{\zeta}_e^{ts}=\begin{cases}e, & if\ e\mid t;\\ 0, & otherwise.\end{cases}$$

Therefore

$$\begin{aligned}\frac{1}{n}\sum_{l=0}^{n-1}\zeta_p^{T(\beta\gamma^{le})}\chi^w(\gamma^{le}) &= \frac{1}{en}\sum_{x\in\mathbb{F}_q^\times}\zeta_p^{T(\beta x)}\chi^w(x)\sum_{s=0}^{e-1}\chi^{ns}(x)\\ &= \frac{1}{en}\sum_{s=0}^{e-1}\sum_{x\in\mathbb{F}_q^\times}\chi^{ns+w}(x)\zeta_p^{T(\beta x)}. \qquad (21)\end{aligned}$$

If $(j-i,\tau)=(0,0)$, we have $1 \le w \le n-1$, $\beta=\gamma^i-\gamma^{j+\tau e}=0$ and $\chi^{ns+w} \neq 1$ for all s $(0 \le s \le e-1)$. Therefore by (20) and (21),

$$\langle \phi_i, \mathbf{M}_w \mathbf{L}_\tau(\phi_j)\rangle = \frac{1}{en}\sum_{s=0}^{e-1}\sum_{x\in\mathbb{F}_q^\times}\chi^{ns+w}(x) = \frac{1}{en}\sum_{s=0}^{e-1} 0 = 0.$$

If $(j-i,\tau)\neq(0,0)$, then $\beta \neq 0$ and the right hand side of (21) is $\frac{1}{en}\sum_{s=0}^{e-1}\bar{\chi}^{ns+w}(\beta)G(\bar{\chi}^{ns+w})$. Therefore

$$\langle \phi_i, \mathbf{M}_w \mathbf{L}_\tau(\phi_j)\rangle \le \frac{1}{en}\sum_{s=0}^{e-1}|G(\bar{\chi}^{ns+w})| \le \frac{e\sqrt{q}}{en} = \frac{\sqrt{en+1}}{n}.$$

The upper bound on λ then follows. ■

Remark 8: For the $(n,M)=((q-1)/e,e)$ time-phase signal set $\mathcal{S}$ in Theorem 20, the bound of Theorem 15 is $\sqrt{\frac{2ne-e-n}{n^2e-n}}$, which is very close to $\sqrt{\frac{en+1}{n^2}} \ge \lambda_{\mathcal{S}}$ when e is mall. Note that we were not able to compute the exact value $\lambda_{\mathcal{S}}$ for the time-phase signal set $\mathcal{S}$ in Theorem 20, and thus unable to tell if it is optimal with respect to some of the bounds described in this paper.

V. Summary and concluding remarks

In this paper, we developed a number of bounds on unit time-phase signal sets which are derived from existing bounds on unit time signal sets. Although the techniques used in establishing Theorems 5 and 9 are simple, they are very useful for developing better bounds on unit time-phase signal sets. These two techniques employ the two one-way bridges with which a unit time-phase signal set $\mathcal{S}$ is converted into the two unit time signal sets $\mathcal{S}_{\mathcal{S}}$ of (9) and $\bar{\mathcal{S}}_{\mathcal{S}}$ of (10). If one sees these bridges, one would immediately obtain the new bounds on unit time-phase signal sets. However, without seeing these bridges, it may be hard to develop bounds on time-phase signal sets even if one is very familiar with the Welch bound and Levenstein bounds. Theorems 5 and 9 are generic and can be employed to obtain more bounds on unit time-phase signal sets from new bounds on unit time signal sets. Thus, one of the main contributions of this paper is the discovery of these two one-way bridges and Theorems 5 and 9.

The two one-way bridges described in (9) and (10) also suggest two ways to construct good time signal sets from a good time-phase signal set. However, it is open how to construct a good time-phase signal set from a given good time signal set.

It is noticed that time-phase signal sets and time signal sets (also called codebooks) are very different, though they all are subsets of $\mathbb{H}_n$. This is because time-phase signal sets consider both the time and phase distortion, while time signal sets take care of only the time distortion. It is much harder to construct good time-phase signal sets. So far no optimal (n,M) time-phase signal set with $M>1$ is known, while a number of optimal time signal sets (codebooks) have been constructed in the literature [4], [6], [7], [8], [15], [29].

An interesting problem is to construct unit time-phase signal sets meeting or almost meeting the bounds on unit time-phase signal sets described in this paper if this is possible. The infinite series of unit time-phase signal sets of Theorem 19 are optimal. This is the first time that an infinite series of optimal time-phase signal sets are constructed. The time-phase signal sets of Theorem 20 are also very good when e is small. The constructions of these optimal and almost optimal time-phase signal sets are another major contribution of this paper. In general, the bounds on unit time-phase signal sets described in this paper should be very good as they are derived from the linear programming bounds on unit time signal sets.

Given any $(n,1,\lambda)$ unit time-phase signal set $\mathcal{S}$ meeting the bound of (11) (meeting also the bound of (12) since $M=1$), the set $\mathcal{S}_{\mathcal{S}}$ defined in (9) is an $(n,n^2,1/\sqrt{n+1})$ signal set meeting the Welch bound. Such $(n,n^2,1/\sqrt{n+1})$ signal sets are called *SIC-POVMs* in quantum information [30], [9], [20]. Algebraic and numerical constructions of such $(n,n^2,1/\sqrt{n+1})$ signal sets are known for small dimensions n [9],

[22]. It is conjectured that SIC-POVMs exist for every dimension n. However, constructing SIC-POVMs seems to be a very hard problem. Therefore, it is also a hard problem to construct $(n, 1, \lambda)$ unit time-phase signal set $\mathcal{S}$ meeting the bound of (11). SIC-POVMs may be used to construct $(n, 1, \lambda)$ unit time-phase signal set $\mathcal{S}$ meeting the bound of (11). It would be worthy to investigate this. Optimal $(n, n^2, 1/\sqrt{n+1})$ time signal sets are also related to mutually unbiased bases, tight frames and line packing in Grassmannian space [5], [2], [24], [8], [4], [6].

Incoherent systems are also related to time signal sets [13], [19]. Bounds on incoherent systems may be employed to derive bounds on time-phase signal sets with the two bridges established in this paper.

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